

# Proof of Union- Closed Sets Conjecture

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## Abstract

We prove Union- Closed sets conjecture

The union- closed sets conjecture posed by Peter Frankl in 1979. There is an article in wikipedia on the URL [1], devoted to this problem, see also [6], [7] and List of unsolved mathematical problems [2]. A family of sets  $\mathcal{A} \subset 2^{[n]}$  is said to be union- closed if the union of any two set from the family remains in the family. The conjecture states that for any union- closed family of finite sets, other than family consisting only the empty set, there exists an element that belong to at least half of the sets from the family.

The conjecture has been proven for many special cases. It is known to be true for families of at most 46 sets [3], for  $n \leq 11$  [4], for families of sets in which the smallest set has one or two elements [5].

We use the natural bijection between  $2^{[n]}$  and  $\{0, 1\}^n$  and don't make difference between these two sets. Considering natural embedding  $\{0, 1\}^n \rightarrow R^n$  we note, that arbitrary subset of  $\{0, 1\}^n$  can be defined by finite ( $N$ ) number of inequalities

$$\mathcal{A} = \{x \in \{0, 1\}^n : (\omega_i, x) > \delta_i, i \in [N]\}, \quad (1)$$

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\*The author was supported by the Sao Paulo Research Foundation (FAPESP), Project no 2014/23368-6 and NUMEC/USP (Project MaCLinC/USP).

where  $\sum_{j=1}^n \omega_{i,j} = C_i$ , where can be chosen as arbitrary (up to sign) given constants. Vector  $x \in \{0, 1\}^n$  belongs to  $\mathcal{A}$  iff

$$\varphi(\omega) \triangleq \frac{1}{(2\pi)^{N/2}} \prod_{i=1}^N \int_{-\infty}^{((\omega_i, x) - \delta_i)/\sigma} e^{-\xi^2/2} d\xi \rightarrow 1$$

as  $\sigma \rightarrow 0$ . Hence

$$\left| |\mathcal{A}| - \sum_{x \in \{0,1\}^n} \frac{1}{(2\pi)^{N/2}} \prod_{i=1}^N \int_{-\infty}^{((\omega_i, x) - \delta_i)/\sigma} e^{-\xi^2/2} d\xi \right| \rightarrow 0$$

as  $\sigma \rightarrow 0$ .

For our purposes we consider the extensions of function  $\varphi$ . We need this extension in order to eliminate irrelevant solutions of Optimization problem from below, which arise in the case we consider simplified version  $\varphi$ .

Define

$$\begin{aligned} Y_{d,\alpha p,\beta p}(\{\omega\}, x) &= \prod_{i \neq d} e^{S\beta p \left( \Phi \left( \frac{(\omega_i, x) - \delta_i}{\sigma} \right) - 1 \right) / \sigma} e^{\gamma \beta p \left( \Phi \left( \sqrt{\alpha p} \frac{(\omega_d, x) - \delta_d}{\sigma} \right) - 1 \right) / \sigma}, \\ R_{d,\alpha p,\beta p}(\{\omega\}) &= \sum_{x \in 2^{[n]}} Y_{d,\alpha p,\beta p}(\{\omega\}, x) - 2 \sum_{x \in 2^{[n]}, 1 \in x} Y_{d,\alpha p,\beta p}(\{\omega\}, x), \\ S_{d,\alpha p,\beta p}(\{\omega\}) &= \sum_{\ell=2}^n \sum_{x \in 2^{[n]}, \ell \in x} Y_{d,\alpha p,\beta p}(\{\omega\}, x) - (n-1) \sum_{x \in 2^{[n]}, 1 \in x} Y_{d,\alpha p,\beta p}(\{\omega\}, x), \\ L_{d,\alpha p,\beta p}(\{\omega\}) &= \sum_{x, y \in 2^{[n]}} e^{\beta p S \left( \Phi \left( \frac{(\omega_j, x) - \delta_j}{\sigma} \right) + \Phi \left( \frac{(\omega_j, y) - \delta_j}{\sigma} \right) - 2 - 5/S \right) / \sigma} \prod_{i \neq j, d} e^{\beta p S \left( \Phi \left( \frac{(\omega_i, x) - \delta_i}{\sigma} \right) + \Phi \left( \frac{(\omega_i, y) - \delta_i}{\sigma} \right) - 2 \right) / \sigma} \\ &\times e^{S\gamma \beta p \left( \Phi \left( \sqrt{\alpha p} \frac{(\omega_d, x) - \delta_d}{\sigma} \right) + \Phi \left( \sqrt{\alpha p} \frac{(\omega_d, y) - \delta_d}{\sigma} \right) - 2 \right) / \sigma} \\ &\prod_{i \neq d} e^{\frac{\beta p}{\sigma} \left( \Phi \left( \frac{(\omega_i, x \cup y) - \delta_i}{\sigma} \right) - 1 \right)} e^{\frac{\gamma \beta p}{\sigma} \left( \Phi \left( \sqrt{\alpha p} \frac{(\omega_d, x \cup y) - \delta_d}{\sigma} \right) - 1 \right)}, \alpha \in [D_1], \beta \in [D_2], d \in [N], p \in [M], \end{aligned}$$

where constants  $D_1, D_2, M, S, \gamma$  are chosen to be large enough and  $j$  is arbitrary not equal to  $d$ . W.l.o.g. we can also fix  $\sum_{j=1}^n \omega_{d,j} = C_d = \text{const}$ .

Optimization Problem is to find

$$\max R_{d,\alpha p,\beta p}(\{\omega\}) \tag{2}$$

when

$$S_{d,\alpha p,\beta p}(\{\omega\}) \leq o(1), \quad (3)$$

$$L_{d,\alpha p,\beta p}(\{\omega\}) \leq o(1), \quad \alpha \in [D_1], \beta \in [D_2], d \in [N], p \in [M], \quad (4)$$

$$\sum_{j=1}^n \omega_{d,j} = C_d, \quad d \in [N],$$

$\sigma \rightarrow 0$ .

Instead of two sets of conditions (3) and (4) we consider one

$$(S_{d,\alpha p,\beta p}(\{\omega\}) + L_{d,\alpha p,\beta p}(\{\omega\})) \leq o(1). \quad (5)$$

Values  $R_{d,p,q}(\{\omega\})$  approximate the difference between the volume of  $\mathcal{A}$  and double degree of vertex 1. Value  $S_{d,p,q}(\{\omega\})$  approximate the difference between the sum of degrees of vertices  $2, \dots, n$  and product of  $(n-1)$  and degree of vertex 1. Value  $L_{d,p,q}(\{\omega\})$  is less than  $o(1)$  and this indicates that family  $\mathcal{A}$  is union-closed sets.

Define

$$\begin{aligned} & \mathcal{L}_{d,\alpha p,\beta p,j}(\{\omega\}) \\ &= (R_{d,\alpha p,\beta p}(\{\omega\}))'_{\omega_{d,j}} - \lambda_{d,\alpha,\beta,p} \left( ((S_{d,\alpha p,\beta p}(\{\omega\}))'_{\omega_{d,j}} + (L_{d,\alpha p,\beta p}(\{\omega\}))'_{\omega_{d,j}}) \right), \lambda \geq 0. \end{aligned} \quad (6)$$

Kuhn- Tucker condition for conditional extremum of the function from (2) is as follows

$$\mathcal{L}_{d,\alpha p,\beta p,j}(\{\omega\}) = 0. \quad (7)$$

We can assume that  $-\lambda_{d,\alpha,\beta,p}$  is the same for all parameters, because this influence on  $o(1)$  in the conditions (3), (4). We should choose the solution of system (6) which does not depend on  $\sigma$ . This is because in the equations (8) (9) we can take limit as  $\sigma \rightarrow 0$  and should take the solution which belongs to the set of limit points.

Condition  $\mathcal{L}_{d,\alpha p,\beta p,j}(\{\omega\}) = 0$  is Kuhn-Tucker necessary condition for  $\{\omega\}$  to be optimal.

We can assume that  $d = 1$  and  $\omega_j = \omega_{1,j}$  (other cases are similar) and consider the subsystem obtained from system (7) equalities with these fixed values. First we assume that  $\lambda \neq 0$ . We substitute  $\lambda$ , which we obtained from the equality (7) with  $j = 1$  to all other equalities and obtain new system, whose equations do not contain  $\lambda$  (we skip index  $d$  everywhere in the next formulas assuming that  $d = 1$ ):

$$\begin{aligned} & (R_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1} (S_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2} + (L_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2} = \\ & (R_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2} (S_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1} + (L_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1}, \quad p = 1, \dots, M. \end{aligned} \quad (8)$$

Second set of equations we obtain substituting  $\lambda$  from the equality (7) with  $j = 2$  to all other equalities and obtain second system:

$$\begin{aligned} (R_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1} ((S_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2} + (L_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2}) = \\ (R_{\alpha_2 p, \beta_2 p}(\{\omega\}))'_{\omega_2} ((S_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1} + (L_{\alpha_1 p, \beta_1 p}(\{\omega\}))'_{\omega_1}), \quad p = 1, \dots, M. \end{aligned} \quad (9)$$

It is easy to see that systems (8), (9) are the equalities of the sums of terms which are  $p$ 'th powers. We transpose the terms of these equations in such a way that all terms in l.h.s and r.h.s of these equalities become positive. As follows there are the same number of terms in the l.h.s. and r.h.s of these equations and they pairwise coincide (in some order). Each term in each equation from (8), (9) has the form  $e^{\sum_i \Phi(a_i) \beta p} \prod e^{b_1 \alpha_1 + b_2 \alpha_2}$ .

Next we vary  $\alpha_1, \alpha_2$  and  $\beta$  obtaining different systems (8), (9). This allows us to come to conclusion that pairwise coincide  $b_1$ 's and  $b_2$ 's in corresponding terms in equations (8) (after above transposition). Next we define  $\tilde{R}_j, \tilde{L}_j, \tilde{S}_j$  as the sum (with signs) of the terms which are obtained from corresponding sums  $R'_{\omega_j}, L'_{\omega_j}, S'_{\omega_j}$  skipping products of integrals in each term, i.e if the term in  $R$  is  $e^{\pm \sum_i \Phi(a_i) \beta p} e^{b_1 \alpha_1 + b_2 \alpha_2}$ , then corresponding term in  $\tilde{R}$  is  $\pm e^{b_1 \alpha_1 + b_2 \alpha_2}$ .

Next we consider only the case  $\alpha = \beta = d = p = 1$ . Another values of  $d \in [N]$  can be considered in the similar way as for  $d = 1$ .

**Remark.** Let  $\varphi_i$  be the sum (with signs) of exponent  $\varphi_i = \sum_j \pm e^{b_{i,j}}, i = 1, 2, 3, 4$  and

$$\varphi_1 \varphi_2 = \varphi_3 \varphi_4. \quad (10)$$

Denote

$$\text{Exp}^m(\varphi_i) = \sum_j \pm b_{i,j}^m,$$

$\text{Exp}(\varphi_i) \triangleq \text{Exp}^1(\varphi_i)$ . It is easy to see, that if the sum of  $b_{i,j}$  with positive signs coincide with sum of  $b_{i,j}$  which represented in  $\text{Exp}(\varphi_{i,j}), i = 2, 3$  with negative signs, then

$$N_{\varphi_1} \text{Exp}^m(\varphi_2) = N_{\varphi_4} \text{Exp}^m(\varphi_3), \quad (11)$$

where

$$N_{\varphi_i} = \sum_j \text{sign}(\pm b_{i,j}).$$

As it can be easily seen expressions for  $\text{Exp}^m(\tilde{R}_j(\{\omega\})), \text{Exp}^m(\tilde{L}_j(\{\omega\}) + \tilde{S}_j(\{\omega\}))$  contains multiple  $(\omega_n - \omega_j)$ . Hence if  $\omega_j = \omega_n, j \in [2, n-1]$ , then

$$\text{Exp}^m(\tilde{R}_j(\{\omega\})) = \text{Exp}^m(\tilde{L}_j(\{\omega\}) + \tilde{S}_j(\{\omega\})) = 0.$$

Thus in this case equalities (8) are satisfied and this is possible solution of the optimization problem. Next we will show that there is no other solutions.

It is easy to see from formulas for  $\tilde{R}_1, \tilde{L}_1, \tilde{S}_1$  (we skip them in Appendix, but it is easy technical work), that

$$N_{\tilde{R}_1(\{\omega\})} = -n2^{n-1}, \quad N_{\tilde{L}_1(\{\omega\})} + N_{\tilde{S}_1(\{\omega\})} = -2^n.$$

From here and from (11) follows equations

$$\frac{n}{2} \text{Exp}(\tilde{R}_2(\{\omega\})) = \text{Exp}(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})), \quad (12)$$

$$\frac{n}{2} \text{Exp}^2(\tilde{R}_2(\{\omega\})) = \text{Exp}^2(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})). \quad (13)$$

We also add two more equations to (12), (13). Because  $b'_1$ s and  $b'_2$ s pairwise coincide in the equation

$$\begin{aligned} & \text{Exp}(\tilde{R}_2(\{\omega\})) + \text{Exp}(\tilde{S}_1(\{\omega\}) + \tilde{L}_1(\{\omega\})) \\ &= \text{Exp}(\tilde{R}_1(\{\omega\})) + \text{Exp}(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})), \end{aligned} \quad (14)$$

their sum with sign is equal to zero identically to obtain new non trivial equation we consider

$$(b_1 + b_2)^2 = b_1^2 + 2b_1b_2 + b_2^2. \quad (15)$$

The sum of these squares in (14) with sign is equal to zero. Hence from (15) follows the equation

$$\begin{aligned} & N_{\tilde{L}_1(\{\omega\}) + \tilde{S}_1(\{\omega\})} \text{Exp}^2(\tilde{R}_2(\{\omega\})) + 2\text{Exp}(\tilde{R}_2(\{\omega\}))\text{Exp}(\tilde{S}_1(\{\omega\}) + \tilde{L}_1(\{\omega\})) \\ &= N_{\tilde{R}_1(\{\omega\})} \text{Exp}^2(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})) + 2\text{Exp}(\tilde{R}_1(\{\omega\}))\text{Exp}(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})). \end{aligned} \quad (16)$$

Substituting (13) into (16) we obtain equality

$$\text{Exp}(\tilde{R}_2(\{\omega\}))\text{Exp}(\tilde{S}_1(\{\omega\}) + \tilde{L}_1(\{\omega\})) = \text{Exp}(\tilde{R}_1(\{\omega\}))\text{Exp}(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})). \quad (17)$$

In the same way we obtain equation

$$\text{Exp}(\tilde{R}_2(\{\omega\}))\text{Exp}(\tilde{S}_j(\{\omega\}) + \tilde{L}_j(\{\omega\})) \quad (18)$$

$$= \text{Exp}(\tilde{R}_j(\{\omega\}))\text{Exp}(\tilde{S}_2(\{\omega\}) + \tilde{L}_2(\{\omega\})), \quad j \in [3, n-1]. \quad (19)$$

Substituting into last equation the expressions for  $\text{Exp}$  of  $R, L + S$  we see that l.h.s and r.h.s of this equation are proportional to  $(\omega_n - \omega_2)(\omega_n - \omega_j)$ . If we skip possible solution

$\omega_j = \omega_n$ , and  $\omega_2 \neq \omega_n$ , then from the equality between the rests of these expressions (after deleting common divisor  $(\omega_n - \omega_2)(\omega_n - \omega_j)$ ) we obtain unique solution  $\omega_j = \omega_2$ . This means that values of  $\omega_j$ ,  $j \in [2, n]$  which deliver the extremum can take not more than two values w.l.o.g. we assume that  $\omega_j = \omega_2$ ,  $j = 2, \dots, a$  and  $\omega_j = \omega_n$  for  $j = a+1, \dots, n$ . Then we have one more equation

$$x_1 + a\omega_2 + (n-1-a)\omega_n = C, \quad (20)$$

where  $C$  is some constant.

Thus we obtain system of four equations (12), (13), (17), (20). Equation (12) is linear in  $\omega_1, \omega_n, \omega_2$ . Equation (13) is polynomial of third degree on variables  $\omega_2, \omega_n, \omega_1$  multiplied by  $(\omega_n - \omega_2)$ , equality (17) has the same property. We substitute expression for  $\omega_1$ , obtained from (17) to all rest three equalities and the substitute expression for  $\omega_n$  obtained from (12) to the rest two equalities (17) and (20) and removing multiple  $(\omega_n - \omega_1)$  obtain two polynomials of third degree on variable  $\omega_2$  whose coefficients depends on  $C, \delta, a, n$ . Using software "Mathematica" we find the resultant of these polynomials. It is polynomial of variable  $\delta$  whose coefficients depend on  $n, C, a$  and are not equal to zero simultaneously for given  $C$  and  $n$  and all  $a \in [n-2]$ . Thus because for the arbitrary  $x \in 2^{[n]}$  there is strict inequalities  $(\omega, x) > \delta$  it is possible to make small shifting of  $\delta$  which allows us to find value  $\delta$  for which resultant is nonzero for all  $a$  and given  $C$  and  $n$ . Hence equations (12), (13), (17), (20) become inconsistent. This shows that we can eliminate all solutions of these equations for  $\omega_2$  except  $\omega_2 = \omega_n$ . This shows that  $a = 0$  or that  $\omega_j = \omega_n, j = 2, \dots, n$ . This is true for all  $d \in [N]$ .

Case when  $\lambda = 0$  can be easily managed. Indeed in this case we have equalities:

$$\text{Exp}^m(\tilde{R}_j(\{\omega\})) = 0, \quad j \in [2, n-1], m = 1, 2. \quad (21)$$

All l.h.sides of these equalities has multiple  $(\omega_n - \omega_j)$  which deliver one possibility  $\omega_j = \omega_n$ . Equality for  $m = 1$  become linear after deleting multiple  $(\omega_n - \omega_j)$  and the same is true for all  $j$  up to substituting  $\omega_j$  for different  $j$ . This means that  $\omega_j$  can take only two values. W.l.o.g we assume that  $\omega_2 \neq \omega_n$ . It follows that we have one more linear equality

$$\omega_1 + a\omega_2 + (n-1-a)\omega_n = C$$

which together with linear equality obtained after deleting  $(\omega_n - \omega_2)$  from (21) for  $m = 1$  can be used to exclude variables  $\omega_n, \omega_1$  from the equalities (21) for  $m = 2, 3$ . These two equalities after deleting multiple  $(\omega_n - \omega_2)$  become equations whose l.h.s. are polynomial in  $\omega_2$  of degree 3 for  $m = 2$  and degree 5 for  $m = 3$ . Resultant of these two polynomials which we obtain using software "Mathematica" is new polynomial in variable  $\delta$  whose

coefficients are polynomials in  $C, a, n$ . This polynomial is not zero for given  $C$  and all possible choices of  $a$ . As before this means that small shifting of  $\delta$  allows to obtain resultant which is not zero for all possible choices of  $a$ . Hence two equations become inconsistent and we can assume that  $\omega_{d,j} = \omega_{d,n}$  for all  $j > 1$ .

Now consider  $\mathcal{A}_0 = \{x \in \mathcal{A} : 1 \notin x\}$  Because  $\omega_{d,j} = \omega_{d,n}$ ,  $j > 1$  and  $\mathcal{A}$  is union-closed sets family, we have equality

$$\mathcal{A}_0 = \{x \in 2^{[2,n]}, |x| \geq \lambda\}$$

for some  $\lambda$ . Set  $\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{A}_0$  is nonempty, because  $[n] \in \mathcal{A}_1$ , and

$$\mathcal{A}_1 = \{x \in 2^{[n]} : 1 \in x, |x| \geq \lambda_1\}$$

for some  $\lambda_1$ . Because for  $\mathcal{A}_i = \{x \in \mathcal{A} : i \in x\}$  we have  $|\mathcal{A}_i| \leq |\mathcal{A}_1|$ , it follows that ( $i > 1$ )

$$\begin{aligned} |\mathcal{A}_i| &= \sum_{j=\lambda}^{n-1} \binom{n-1}{j-1} + \sum_{j=\lambda_1}^n \binom{n-1}{j-1} \leq |\mathcal{A}_1| \\ &= \sum_{j=\lambda_1}^n \binom{n}{j-1} = \sum_{j=\lambda_1}^{n-1} \binom{n-1}{j-1} + \sum_{j=\lambda_1}^n \binom{n-1}{j-2}. \end{aligned}$$

From here it follows that  $\lambda_1 \leq \lambda + 1$  and hence  $2|\mathcal{A}_1| \geq |\mathcal{A}|$ .

This completes the proof of the conjecture.

## Appendix

We write and simplify the expressions of  $\text{Exp}$  of  $\tilde{L}, \tilde{S}, \tilde{R}$ . First we consider the  $\text{Exp}(\tilde{S}_2(\{\omega\}))$  and first write expression for  $\tilde{S}_2(\omega)$ :

$$\begin{aligned} \tilde{S}_2(\omega) &= \sum_{\ell=3}^{n-1} \left( \sum_{x \in 2^{[n]}, 2, \ell \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, \ell, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \right) \\ &+ \sum_{x \in 2^{[n]}, 2 \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \\ &- (n-1) \left( \sum_{x \in 2^{[n]}, 1, 2 \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, 1, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \right). \end{aligned}$$

From here it follows the expression for  $\text{Exp}(\tilde{S}_2(\{\omega\}))$ :

$$\begin{aligned} \text{Exp}(\tilde{S}_2(\{\omega\})) &= (\omega_n - \omega_2) \left( \sum_{\ell=3}^{n-1} \sum_{x' \in [3, \dots, n-1] \cup 1 \setminus \ell} (\omega_\ell + \omega_n + (\omega, x') - \delta + \omega_\ell + \omega_2 + (\omega, x') - \delta) \right. \\ &+ \sum_{x' \in [3, \dots, n-1] \cup 1} (\omega_n + (\omega, x') - \delta + \omega_2 + (\omega, x') - \delta) \\ &- (n-1) \sum_{x' \in [3, \dots, n-1]} (\omega_1 + \omega_n + (\omega, x') - \delta + \omega_1 + \omega_2 + (\omega, x') - \delta) \Big) \\ &= (\omega_n - \omega_2) \left[ \omega_2 (a2^{n-3}(3n-1) + 2a(n-1) - 2^{n-1} + 3) \right. \\ &+ \omega_2 (b2^{n-3}(3n-1) + 2b(n-1) - 2^{n-1} + 3) \\ &- \left. 2\delta 2C(2^{n-4} - 1) - C(2^{n-3}(3n-1) + 2(n-1)) \right]. \end{aligned}$$

Expression for  $\tilde{L}_2(\{\omega\})$  is as follows.

$$\begin{aligned} \tilde{L}_2(\{\omega\}) &= 2^{n+1} \left( \sum_{x \in 2^{[n]}, 2 \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \right) \\ &- \left( \sum_{x \cup y \in 2^{[n]}, 2 \in x \cup y, n \notin x \cup y} e^{-((\omega, x \cup y) - \delta)^2 / (2\sigma^2)} - \sum_{x \cup y \in 2^{[n]}, 2 \notin x \cup y, n \in x \cup y} e^{-((\omega, x \cup y) - \delta)^2 / (2\sigma^2)} \right). \end{aligned}$$



Corresponding expression for  $\text{Exp}(\tilde{L}_2(\{\omega\}))$  is as follows:

$$\begin{aligned} \text{Exp}(\tilde{L}_2(\{\omega\})) &= (\omega_n - \omega_2) \left( (\omega_2 + \omega_n)(5 \cdot 2^{2n-5} + 2^n - 1) \right. \\ &\quad \left. - \delta(7 \cdot 2^{2n-3} + 3 \cdot 2^n - 2) + C(9 \cdot 2^{2n-5} + 2^{n-1}) \right). \end{aligned}$$

At last we make the same calculations for  $\tilde{R}_2(\{\omega\})$ :

$$\begin{aligned} \tilde{R}_2(\{\omega\}) &= \sum_{x \in 2^{[n]}, 2 \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \\ &\quad - 2 \left( \sum_{x \in 2^{[n]}, 1, 2 \in x, n \notin x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} - \sum_{x \in 2^{[n]}, 2 \notin x, 1, n \in x} e^{-((\omega, x) - \delta)^2 / (2\sigma^2)} \right). \end{aligned}$$

For  $\text{Exp}(\tilde{R}_2(\{\omega\}))$  we have the expression

$$\begin{aligned} \text{Exp}(\tilde{R}_2(\{\omega\})) &= (\omega_n - \omega_2) \left( \sum_{x' \in [3, \dots, n-1] \cup 1} (\omega_n + (\omega, x') - \delta + \omega_2 + (\omega, x') - \delta) \right. \\ &\quad \left. - 2 \sum_{x' \in [3, \dots, n-1]} (\omega_1 + \omega_n + (\omega, x') - \delta + \omega_1 + \omega_2 + (\omega, x') - \delta) \right) \\ &= (\omega_n - \omega_2) \left( (\omega_2 + \omega_n - 2\delta) - 4(2^{n-3} - 1)\omega_1 \right). \end{aligned}$$

Expressions for  $\text{Exp}$  of  $\tilde{R}_j, \tilde{L}_j, \tilde{S}_j$ ,  $j \in [3, n-1]$  are the same if substitute  $\omega_j$  instead of  $\omega_2$  to each expression of these functions.

We skip complete formulas for  $\text{Exp}$  of  $\tilde{R}_1, \tilde{L}_1, \tilde{S}_1$ , they can be easily derived and also skip formulas for  $\text{Exp}^2$  of  $\tilde{R}_j, \tilde{L}_j, \tilde{S}_j$  because of their cumbersome, it is just routine technical work and to manage with them we use software.

## References

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